PHILOSOPHICAL TRANSACTIONS.

I. *Supplement to a Paper "On the Theoretical Explanation of an apparent new Polarity in Light." By* G. B. AIRY, *Esq. M.A. F.R.S., Astronomer Royal.*

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IN the Second Part of the Transactions of the year 1840, the Royal Society has published a memoir by me, explaining, on the undulatory theory of light, the apparent new polarity observed by Sir DAVID BREWSTER; which explanation is based upon the assumption that the spectrum is viewed out of focus; an assumption which corresponded to the circumstances of my own observations, and to those of some other persons. Since the publication of that memoir, I have been assured by Sir DAVID BREWSTER that the phenomenon was most certainly observed with great distinctness when the spectrum was viewed so accurately in focus that many of FRAUNHOFER's finer lines could be seen. This observation appeared to be contradictory to those of Mr. TALBOT, cited by me in page 226 of the memoir, as well as to my own. With the view of removing the obscurity that still appeared to embarrass this subject, I have continued the theoretical investigation for that case which was omitted in the former memoir, namely, when the spectrum is viewed in focus, or when $a = 0$ (page 229); and I have arrived at a result which appears completely to reconcile the seemingly conflicting statements.

In the following investigation I shall use the symbols and the formulæ of the former memoir (as far as they apply) without further reference.

The value of ϱ in page 228 becomes, on making $a = 0$,

$$\varrho = e - \frac{b}{e} y,$$

and the disturbance of ether, on the point of the retina whose distance from the geometrical image is b, produced by a small portion δy of the front of the wave, is

$$\delta y \times \sin \frac{2\pi}{\lambda} (v t - \varrho)$$

or

$$\delta y \times \sin \frac{2\pi}{\lambda} \left(v t - e + \frac{b}{e} y\right),$$

and therefore the whole disturbance of ether on the point of the retina, produced by that part of the pupil which is not covered by any retarding plate, is

$$\int_y \sin \frac{2\pi}{\lambda}\left(v t - e + \frac{b}{e} y\right),$$

the limits of the integral being the values of y corresponding to the boundaries of the part of the pupil not covered by a retarding plate.

But if a portion of the pupil be covered by a plate producing the retardation R (expressed as an angle) in the phase of the wave, the expression to be integrated through the limits proper for the covered part will be

$$\int_y \sin\left(\frac{2\pi}{\lambda}(v t - \varrho) - \mathrm{R}\right)$$

or

$$\int_y \sin\left(\frac{2\pi}{\lambda}\left(v t - e + \frac{b}{e} y\right) - \mathrm{R}\right).$$

Let the limits of the pupil be from $-h$ to $+h$, without regard to the other ordinate upon its surface (which amounts to supposing the form of the pupil to be a parallelogram), and let the part which depends on R be taken between the limits 0 and $+h$ (which amounts to supposing that half of the pupil to be covered which is on the side on which b is considered positive). Then the whole disturbance of the ether is

$$\int_y \sin \frac{2\pi}{\lambda}\left(v t - e + \frac{b}{e} y\right) \text{ from } y = -h \text{ to } y = 0$$

$$+\int_y \sin\left(\frac{2\pi}{\lambda}\left(v t - e + \frac{b}{e} y\right) - \mathrm{R}\right) \text{ from } y = 0 \text{ to } y = +h$$

$$= \frac{\lambda e}{2\pi b}\left\{\cos \frac{2\pi}{\lambda}\left(v t - e - \frac{b h}{e}\right) - \cos \frac{2\pi}{\lambda}(v t - e) + \cos\left(\frac{2\pi}{\lambda}(v t - e) - \mathrm{R}\right)\right.$$

$$\left. - \cos\left(\frac{2\pi}{\lambda}\left(v t - e + \frac{b h}{e}\right) - \mathrm{R}\right)\right\}.$$

The coefficient of $\cos \frac{2\pi}{\lambda}(v t - e)$ is

$$\frac{\lambda e}{2\pi b}\left\{\cos \frac{2\pi}{\lambda} \cdot \frac{b h}{e} - 1 + \cos \mathrm{R} - \cos\left(\frac{2\pi}{\lambda} \cdot \frac{b h}{e} - \mathrm{R}\right)\right\}$$

$$= -\frac{\lambda e}{2\pi b}\left\{1 - \cos \frac{2\pi}{\lambda} \cdot \frac{b h}{e} - \cos \mathrm{R} + \cos \frac{2\pi}{\lambda} \cdot \frac{b h}{e} \times \cos \mathrm{R} + \sin \frac{2\pi}{\lambda} \cdot \frac{b h}{e} \times \sin \mathrm{R}\right\}$$

$$= -\frac{\lambda e}{2\pi b}\left\{\left(1 - \cos \frac{2\pi}{\lambda} \cdot \frac{b h}{e}\right) \times (1 - \cos \mathrm{R}) + \sin \frac{2\pi}{\lambda} \cdot \frac{b h}{e} \times \sin \mathrm{R}\right\}$$

$$= -\frac{\lambda e}{2\pi b} 4 \sin \frac{\pi}{\lambda} \cdot \frac{b h}{e} \times \sin \frac{\mathrm{R}}{2} \times \left\{\sin \frac{\pi}{\lambda} \cdot \frac{b h}{e} \times \sin \frac{\mathrm{R}}{2} + \cos \frac{\pi}{\lambda} \cdot \frac{b h}{e} \times \cos \frac{\mathrm{R}}{2}\right\}$$

$$= -\frac{2\lambda e}{\pi b} \cdot \sin \frac{\pi b h}{\lambda e} \cdot \sin \frac{\mathrm{R}}{2} \cdot \cos\left(\frac{\pi b h}{\lambda e} - \frac{\mathrm{R}}{2}\right);$$

and the coefficient of $\sin \frac{2\pi}{\lambda}(v\,t - e)$ is

$$\frac{\lambda e}{2\pi b}\left\{\sin\frac{2\pi}{\lambda}\cdot\frac{b\,h}{e} + \sin \mathrm{R} + \sin\left(\frac{2\pi}{\lambda}\cdot\frac{b\,h}{e} - \mathrm{R}\right)\right\}$$

$$= \frac{\lambda e}{2\pi b}\left\{\sin\frac{2\pi}{\lambda}\cdot\frac{b\,h}{e}\times(1+\cos\mathrm{R}) + \sin\mathrm{R}\times\left(1-\cos\frac{2\pi}{\lambda}\cdot\frac{b\,h}{e}\right)\right\}$$

$$= \frac{2\lambda e}{\pi b}\cdot\sin\frac{\pi b h}{\lambda e}\cdot\cos\frac{\mathrm{R}}{2}\cdot\left\{\cos\frac{\pi b h}{\lambda e}\cdot\cos\frac{\mathrm{R}}{2} + \sin\frac{\mathrm{R}}{2}\cdot\sin\frac{\pi b h}{\lambda e}\right\}$$

$$= \frac{2\lambda e}{\pi b}\cdot\sin\frac{\pi b h}{\lambda e}\cdot\cos\frac{\mathrm{R}}{2}\cdot\cos\left(\frac{\pi b h}{\lambda e} - \frac{\mathrm{R}}{2}\right).$$

And the intensity of light on the point of the retina, which is represented by the sum of the squares of these coefficients, is

$$\frac{4\lambda^2 e^2}{\pi^2 b^2}\cdot\sin^2\frac{\pi b h}{\lambda e}\cdot\cos^2\left(\frac{\pi b h}{\lambda e} - \frac{\mathrm{R}}{2}\right).$$

For convenience, put $\frac{\pi b h}{\lambda e} = w$, and omit the constant factor $4\,h^2$; the expression becomes then

$$\left(\frac{\sin w}{w}\right)^2.\cos^2\left(w - \frac{\mathrm{R}}{2}\right),$$

where it must be borne in mind that w is a multiple of the distance, of the point of the retina at which the intensity is sought, from the geometrical image of the point of light. It must also be borne in mind that this expression gives the intensity on that point of the retina produced by a single point of light, or a single line of light parallel to the bounding edge of the retarding plate.

The following Table contains the values of $\left(\frac{\sin w}{w}\right)^2 \cos^2\left(w - \frac{\mathrm{R}}{2}\right)$ for every 10° of w, and for every 60° of R. In computing them, w has been expressed in degrees: and the last figure of the numbers contained in the Table is the eighth decimal place.

Table of $\left(\frac{\sin w}{w}\right)^2 \cdot \cos^2\left(w - \frac{R}{2}\right)$.

Values of w.	Values of R.					
	0°	60°	120°	180°	240°	300°
− 175°	25	20	8	0	4	17
− 165	229	230	123	17	17	123
− 155	610	738	499	133	6	245
− 145	1050	1553	1286	515	12	280
− 135	1372	2560	2560	1372	184	184
− 125	1413	3527	4261	2882	767	33
− 115	1109	4168	6164	5101	2043	47
− 105	567	4231	7896	7896	4231	567
− 95	84	3618	9032	10912	7378	1964
− 85	104	2453	9216	13631	11282	4519
− 75	1111	1111	8294	15475	15475	8294
− 65	3473	148	6396	15970	19294	13046
− 55	7298	169	3962	14884	22013	18221
− 45	12346	1654	1654	12346	23038	23038
− 35	18020	4796	204	8835	22060	26652
− 25	23474	9402	217	5104	19175	28360
− 15	27778	14887	1994	1994	14886	27778
− 5	30154	20389	5427	231	9996	24959
+ 5	30154	24959	9996	231	5427	20388
+ 15	27778	27778	14887	1994	1994	14887
+ 25	23474	28360	19175	5104	217	9402
+ 35	18020	26651	22060	8835	204	4794
+ 45	12346	23038	23038	12346	1654	1654
+ 55	7298	18220	22013	14884	3962	169
+ 65	3473	13045	19294	15970	6396	148
+ 75	1111	8294	15475	15475	8294	1111
+ 85	104	4519	11282	13631	9216	2453
+ 95	84	1964	7378	10912	9032	3618
+ 105	567	567	4231	7896	7896	4231
+ 115	1109	47	2043	5101	6164	4168
+ 125	1413	33	767	2882	4261	3527
+ 135	1372	184	184	1372	2560	2560
+ 145	1050	279	12	515	1286	1553
+ 155	610	245	6	133	499	738
+ 165	229	123	16	17	123	230
+ 175	25	17	4	0	8	20

It has been deemed unnecessary to continue the Table beyond the values of w — 175° and + 175°, because the values of $\left(\frac{\sin w}{w}\right)^2$ become very small. The greatest maximum of this quantity occurs when $w = 0$; its value (expressing w in terms of the radius) is then $= 1$; the second maximum occurs when $w = \frac{3\pi}{2}$ nearly; its value is then $\left(\frac{2}{3\pi}\right)^2$ nearly $= \frac{1}{22}$ nearly; the amount of which will probably produce an inconsiderable influence on the expressions which we are now about to consider.

The curves in the annexed figure represent, by their ordinates, the values of $\left(\frac{\sin w}{w}\right)^2 \cdot \cos^2\left(w - \frac{R}{2}\right)$; the values of R being continued as far as 720°, in order to exhibit more distinctly to the eye the successive displacements of the principal bows

of the curves. The ordinates, therefore, represent the intensity of light on different points of that small diffused image on the retina which is formed by the light coming from a single point, even when it is seen accurately in focus; the extreme breadth of the image represented in the figure corresponds to 360° of w, or is $= \frac{2\lambda e}{h}$.

If we express the area of each of the curves by summing the ordinates and dividing the sum by thirty-six, we find the following values:

R = 0°, area is represented by 7234
60, area is represented by 7055
120, area is represented by 6696
180, area is represented by 6517
240, area is represented by 6696
300, area is represented by 7055.

I shall proceed now to apply these numbers to the explanation of the phenomena in question.

Light is supposed to be incident on the eye from different points of a spectrum, formed in any way: the characteristic of the spectrum as concerned in the present investigation being, that the order of position of the different colours is the same as the order of the successive values of R.

First. Suppose the value of $\frac{2\lambda e}{h}$ to be small, at least in comparison with the distance between those points of the image of the spectrum in which R has changed by 360°.

1. Let $\frac{2\lambda e}{h}$ be exceedingly small. Since the same form of curve recurs for every change of 360° in R and not oftener, it is evident that the succession of bands (if there are any) in the visible image will depend on the changes of 360° in R. Our supposition, therefore, amounts to this; that the extent of the small diffused image is exceedingly less than the interval between the bands (if there are any). Here it is plain that the formation of the broad bands cannot depend on the inequalities of light in the narrow diffused image, but must depend on the quantity of light in the whole of each narrow diffused image considered as a total light from one point of the spectrum. Now the total light is equal for all points. For, as the intensity of light coming from one luminous point and falling on a point of the retina is represented by $\left(\frac{\sin w}{w}\right)^2 . \cos^2\left(w - \frac{R}{2}\right)$, the whole light coming from that luminous point is $\int_w \left(\frac{\sin w}{w}\right)^2 . \cos^2\left(w - \frac{R}{2}\right)$, the limits of the integral being $\pm\infty$. Now this definite integral is independent of R. For

$$\cos^2\left(w - \frac{R}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos R + \cos R . \cos^2 w + \sin R . \cos w . \sin w,$$

and therefore

$$\left(\frac{\sin w}{w}\right)^2 \cos^2\left(w - \frac{\mathrm{R}}{2}\right) = \left(\frac{1}{2} - \frac{1}{2}\cos \mathrm{R}\right)\left(\frac{\sin w}{w}\right)^2 + \cos \mathrm{R}\left(\frac{\cos w \,.\, \sin w}{w}\right)^2$$
$$+ \sin \mathrm{R} \,.\, \frac{\cos w \,.\, \sin^3 w}{w^2};$$

and the whole intensity of light is represented by

$$\left(\frac{1}{2} - \frac{1}{2}\cos \mathrm{R}\right)\int_w \left(\frac{\sin w}{w}\right)^2 + \cos \mathrm{R}\int_w \left(\frac{\cos w \,.\, \sin w}{w}\right)^2 + \sin \mathrm{R}\int_w \frac{\cos w \,.\, \sin^3 w}{w^2},$$

the limits of integration being $\pm\ \infty$. The last term, changing sign when w changes sign, evidently makes its definite integral $= 0$: the two former may be put in the shape

$$\left(\frac{1}{2} - \frac{1}{2}\cos \mathrm{R}\right)\int_w \left(\frac{\sin w}{w}\right)^2 + \frac{1}{2}\cos \mathrm{R}\int_{2w} \left(\frac{\sin 2w}{2w}\right)^2.$$

If $\int_w \left(\frac{\sin w}{w}\right)^2$ from $-\ \infty$ to $+\ \infty$ be $= \mathrm{S}$, then $\int_{2w} \left(\frac{\sin 2w}{2w}\right)^2$ from $-\ \infty$ to $+\ \infty$ is also $= \mathrm{S}$, and the expression becomes

$$\left(\frac{1}{2} - \frac{1}{2}\cos \mathrm{R}\right) \cdot \mathrm{S} + \frac{1}{2}\cos \mathrm{R} \,.\, \mathrm{S} \quad \text{or} \ \frac{1}{2}\,\mathrm{S},$$

which is independent of R. The total light, therefore, is independent of R, or is equal at all points; and therefore no bands are produced.

2. But if $\frac{2\lambda e}{h}$, though small, is not exceedingly small, the principal impression may be made upon the eye by the central patch of light from each source, included between the values $w = -\ 180°$, $w = +\ 180°$; while those parts of the light which extend beyond the central patch may be in fact aggregated with the central patches of light from the sources at a small distance on each side. And if the amounts in the central patches from different sources are unequal, while the whole amounts from the different sources are equal, it is evident that a bright central patch from one source may be combined with bright detached parts from another source, while a fainter central patch from that second source may be combined with an insignificant detached part from the first source, and thus the whole inequality of light may be double the inequality of the central patches. Now the amount of the light in the central patch, as we have found, is greatest, and represented by 7234, when $\mathrm{R} = 0$ or $= 2\,n\,\pi$, and is least, and is represented by 6517, when $\mathrm{R} = \pi$ or $= \overline{2\,n + 1} \,.\, \pi$. The difference of these is $\frac{1}{10}$th of the whole; and therefore the difference of the whole light on each part of the retina, formed by combining the central patch formed by one source with the detached light formed by another source, will be nearly $\frac{1}{5}$th of the whole. This inequality of light is amply sufficient to form conspicuous bands.

The bars thus formed depend upon nothing but the changes in the value of R: it is wholly indifferent whether R increases or diminishes towards the side on which b

is considered positive; that is, it is indifferent whether the retarding plate is applied on the same side as the red end or the violet end of the spectrum. These appear to be the bars seen by Mr. TALBOT and myself when the spectrum was viewed in focus. They require that $\frac{2\lambda e}{h}$ be not large, that is, that the aperture of the pupil ($2\,h$), or the aperture of the telescope used be not very small; and that the changes of R be not very rapid; that is, that the plate of mica, &c. be thin. These circumstances held in my own experiment. I may add that the dark bands were not black, but merely dusky; as indicated by the numbers above.

Secondly. Suppose the value of $\frac{\lambda e}{h}$ to be comparable with the distance between those points of the image of the spectrum in which R has changed by 360°; for instance, suppose $\frac{\lambda e}{h}$ to be equal to that distance.

1. Let the red end of the external spectrum be on the same side as the retarding plate, that is, on the side on which b is considered positive. Then on the retina the violet end is on that side; or R increases towards the positive side. Let k be the ordinate measured from a fixed point on the retina to the centre of the diffused image of any colour (k being therefore a function of λ), and l the ordinate measured from the same fixed point to the point at which the intensity is to be ascertained; then $k + b = l$, or $b = l - k$, and the intensity produced by any one kind of light is represented by

$$\frac{\sin^2 \frac{\pi h}{\lambda e}(l-k)}{\left\{\frac{\pi h}{\lambda e}(l-k)\right\}^2} \cos^2\left(\frac{\pi h l}{\lambda e} - \frac{\pi h k}{\lambda e} - \frac{\mathrm{R}}{2}\right).$$

The sum of the intensities on one point of the retina produced by all the different kinds of light from the adjacent portions of the spectrum will be found by varying k in this expression, and adding together all the values so produced. Now if R increases when k increases (as occurs when the red end of the external spectrum is on the same side as the retarding plate), the last factor $\cos^2\left(\frac{\pi h l}{\lambda e} - \frac{\pi h k}{\lambda e} - \frac{\mathrm{R}}{2}\right)$ will undergo very great changes from the combined changes of $\frac{\pi h k}{\lambda e}$ and $\frac{\mathrm{R}}{2}$, whatever be the value of l, and the succession of values which it receives will not differ materially for different values of l; the first factor $\frac{\sin^2 \frac{\pi h}{\lambda e}(l-k)}{\left\{\frac{\pi h}{\lambda e}(l-k)\right\}^2}$ will also undergo great changes, but nearly the same for different values of l; and in consequence the aggregate of all the values for different values of k, exhibiting the total intensity of light upon the point l, will be nearly the same.

This aggregation will be represented graphically by supposing the second curve in

the diagram to be moved towards the right hand, the third to be moved further to the right, &c., and taking the sum of the ordinates of the various curves which are then placed vertically one below the other; it is clear that the large ordinates of one curve will be added to the small ones of another, so as to produce in every part an approximate mean value. If we perform the same operation numerically, combining the last number of the first column in the Table with the last but three in the second column, the last but six in the third column, and so on, to the twelfth column (observing that the numbers in the columns recur after the sixth, or that they may be supposed to recur before the first), and if we remark that by adding the numbers from twelve columns we do in fact combine the intensities from all the diffused images that are in any degree superposed; and if we then divide by twelve, we find the following numbers to represent the intensities:

6884, 6882, 6881, 6879, 6875, 6872, 6870, 6868, 6867, 6868, &c.,

the greatest number being 6884 and the least 6868. It is plain that no bands will be visible here.

2. Let the violet end of the external spectrum be on the same side as the retarding plate. The same algebraic expression holds as in the other case, but there is this important difference in the interpretation, that R (which increases towards the violet end of the spectrum) is greatest in the spectrum on the retina on that side on which k is negative, or when k increases R diminishes. And if $\frac{\lambda e}{h}$ be equal to the change of k corresponding to a change of 2π in R, or if $\frac{\pi h}{\lambda e} \cdot \frac{\lambda e}{h}$ (or π) be equal to the change of $\frac{\pi h k}{\lambda e}$ corresponding to a change of 2π in R, or of π in $\frac{R}{2}$; then the changes of $\frac{\pi h k}{\lambda e}$ and of $\frac{R}{2}$ exactly destroy each other; $\frac{\pi h k}{\lambda e} + \frac{R}{2} =$ a constant C, and the whole intensity of light on a given point will be found by aggregating all the quantities

$$\frac{\sin^2 \frac{\pi h}{\lambda e}(l-k)}{\left\{\frac{\pi h}{\lambda e}(l-k)\right\}^2} \cdot \cos^2\left(\frac{\pi h l}{\lambda e} - C\right),$$

giving different values to k. As the second factor is independent of k, and as the changes of the first caused by changing the values of k will be similar (to the extent to which the light is sensible), whatever be the value of l, it follows that the aggregate will be expressed by the form $B \cos^2\left(\frac{\pi h l}{\lambda e} - C\right)$. This expression denotes that there will be light of all degrees of intensity from the brightest B to zero or total darkness; and that the whole of the changes will recur (or the dark bands will recur) when $\frac{\pi h l}{\lambda e}$ has changed by 2π, or when l has changed by $\frac{2\lambda e}{h}$.

This combination will be represented graphically by drawing back the second curve

of the diagram by 30°, the third by 60°, and so on; and taking the sum of the ordinates which are then vertically one below the other. It is evident that the ordinates *zero* correspond throughout. If we perform the same operation numerically, combining the first number of the first column in the table with the fourth number of the second column, the seventh number of the third column, &c., and if we then divide the sum by twelve, we find the following numbers:

13646, 12829, 11295, 9227, 6875, 4524, 2456, 921, 105, 105, 921, &c.;

the greatest number being 13646, or a little greater, and the least being 0.

It is evident that these numbers denote the formation of most vivid black and bright bands.

The case which we have taken $\left(\text{when } \frac{\lambda e}{h} \text{ is exactly equal to the change of } k \text{ corresponding to a change of } 2\pi \text{ in R}\right)$ is the most favourable for the production of bands; but it will easily be understood that, in consequence of the small extent of the diffused image, conspicuous bands may be formed when the change of k corresponding to a change of 2π in R is sensibly greater or less than $\frac{\lambda e}{h}$.

The interval between the bands is $\frac{2\lambda e}{h}$, and is, therefore, usually small. They will, however, be made broader by making h small, that is, by contracting the aperture of the pupil, or by using a telescope with a limited object-glass. The value of R changes through 2π with no greater change in the quality of light than that produced by passing from one part of the spectrum to another part distant (on the retina) by $\frac{2\lambda e}{h}$, and therefore the retarding plate must be comparatively thick.

It is evident that these are the bands seen by Sir David Brewster when the spectrum was viewed in focus.

The investigation, as regards the explanation of the formation or non-formation of bands under different circumstances, when a thin plate of a transparent medium is placed to cover a portion of the pupil, and the eye is turned to view a spectrum, may now be considered as sufficiently complete, and (I conceive) as perfectly satisfactory. Some change in the expressions would undoubtedly be produced by introducing the consideration of the circular form of the pupil, the inclined position of the transparent plate adopted by Sir David Brewster in some experiments, &c., but none, I apprehend, which would at all affect the general explanation.

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Oct. 23, 1840.

Curves representing by their ordinates the values of $\left(\frac{\sin w}{w}\right)^2 \cos^2\left(w - \frac{R}{2}\right)$.